Math 53, Discussions 116 and 118 **2022-10-07**

More tangent planes, and some review

Questions

Problem 1. Last time, I asked you to find the tangent plane to the surface

$$xy + yz + zx = 5$$

at the point $(1, 2, 1)$. Do it again, but using the method you learned yesterday in lecture.

Problem 2. Let S be the cone $x^2 + y^2 = z^2$ and let H be the plane $x - 2y + 3z = 13$. The curve of intersection $C = S \cap H$ is an ellipse, and the point $P(4, 3, 5)$ is on this ellipse.

(a) Find the tangent plane to S at the point P.
(b) The plane from (a) and the plane H intersect in a line. Parametrize this line.
(c) Find the two possible unit tangents $\mathbf{T}$ to the curve C at the point P.
(d) Find the unit normal $\mathbf{N}$ to the curve C at the point P. This one is conceptually tricky. Here are some observations to help you. The curve C is contained in the plane H, so $\mathbf{N}$ must be parallel to this plane. Also, $\mathbf{N}$ is orthogonal to $\mathbf{T}$, and it points in the direction the curve is "turning."

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1. Letting $F(x, y, z) = xy + yz + zx$, the surface in question is a level set of F. Hence we can use ∇F to find a normal vector at the point $(1, 2, 1)$. The gradient is

$$\nabla F(x, y, z) = \langle y + z, x + z, x + y\rangle$$

and so

$$\nabla F(1, 2, 1) = \langle 3, 2, 3\rangle$$

will suffice as a normal vector. Hence the plane equation is

$$3(x - 1) + 2(y - 2) + 3(z - 1) = 0$$

which is the same equation as last time.

Question 2.

(a) Let $F(x, y, z) = x^2 + y^2 - z^2$ so that the equation of S is $F(x, y, z) = 0$. Then the tangent plane at the point P has normal vector $\nabla F(P) = \langle 8, 6, -10\rangle$. We can harmlessly rescale this vector to $\langle 4, 3, -5\rangle$ and write the equation

$$4(x - 4) + 3(x - 3) - 5(x - 5) = 0.$$

(b) We already know that $(4, 3, 5)$ is on the line, so it suffices to find a direction vector $\mathbf{v}$. Because this line is contained in the plane $x - 2y + 3z = 13$ as well as the plane from the preceding part, it follows that $\mathbf{v}$ ought to be orthogonal to both $\langle 1, -2, 3\rangle$ and $\langle 4, 3, -5\rangle$. So let's take

$$\mathbf{v} = \langle 1, -2, 3\rangle \times \langle 4, 3, -5\rangle = \langle 1, 17, 11\rangle.$$

Our line is then given by

$$\mathbf{r}(t) = \langle 4, 3, 5\rangle + t\langle 1, 17, 11\rangle.$$

(c) The two possible unit tangents are

$$\mathbf{T} = \pm\frac{\mathbf{v}}{\|\mathbf{v}\|} = \pm\frac{\langle 1, 17, 11\rangle}{\sqrt{411}}.$$

(d) The vector $\mathbf{N}$ is parallel to the plane $x - 2y + 3z = 13$ (so, orthogonal to the vector $\langle 1, -2, 3\rangle$) and also orthogonal to the vector $\mathbf{T}$ (or the vector $\mathbf{v}$). So we can start by producing a vector in the correct direction via the cross product:

$$\langle 1, -2, 3\rangle \times \langle 1, 17, 11\rangle = \langle -73, -8, 19\rangle.$$

Next we rescale to a unit vector:

$$\frac{\langle -73, -8, 19\rangle}{\sqrt{5754}}.$$

This vector is either $\mathbf{N}$ or its negative. To see which one it is, it's necessary to examine the picture: we see that $\mathbf{N}$ should point "inwards" towards the z-axis, as that's the direction the ellipse is turning. Since P has coordinates $(3, 4, 5)$, the vector above is pointing in roughly the expected direction. So it is correct.